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THE OPTIMALITY CONDITIONS FOR AN UPPER BOUND ON THE DISPLACEMENT OF A CREEPING BODY SUBJECTED TO VARIABLE LOADING

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Abstract—A displacement bound for a creeping structure subject to variable thermal and mechanical loading was derived in a previous paper[1]. Here we discuss a sufficient condition for the bound to be optimal and suggest simple methods by which near optimal bounds may be computed. The theory is illustrated by evaluation of the bound for a beam problem with a single redundancy.

1. INTRODUCTION

The analysis of the deformations induced in a structure due to the presence of creep deformation and subject to variable thermal and mechanical loading remains amongst the more difficult problems of structural analysis. Analytic solutions are possible for only a very restricted class of problems and numerical solutions, although allowing a wider range of solutions, are still restricted to fairly simple circumstances. The difficulties stem from the inherent non-linearity of the continuum problem and the lack of effective material models.

In this paper we are concerned with the former of these two difficulties and we consider the simplest of the available material models, the elastic/time hardening creep model. For this material we consider a structure subjected to a cyclic history of mechanical and thermal loading. In previous papers[3, 4] a method was described by which bounds may be computed on the displacement of such structures in terms of an equilibrium stress field and a "dummy" load. The result provided a generalization of earlier work by Martin[2] and Leckie and Martin[1]. In this paper we derive sufficient conditions that this bound shall be optimal, with respect to the stress field and dummy load. We find that the problem has a close relationship with the optimal work bounds discussed in reference[5], which was generalized to include thermal loading in[3]. We find that the computation of the optimal bound with respect to the stress field involves a structural problem comparable in complexity with the "steady state" solution. The computation of the optimal bound with respect to the dummy load presents greater difficulty but it is shown in a simple example that in some circumstances at least, a sufficiently accurate approximate value is possible.

In Section 2 the material model and the displacement bound are reviewed and in Section 3 the results for the work bounds are reviewed. In Section 4 the optimality conditions are derived and in Section 5 a simple example is presented.

2. MATERIAL CHARACTERISTICS AND THE DISPLACEMENT BOUND

Consider an elastic/creeping material satisfying the constitutive relationship

$$\underset{\sim}{\epsilon} = \underset{\sim}{e} + \underset{\sim}{v} \tag{1a}$$

$$\underset{\sim}{e} = \underset{\sim}{C}\underset{\sim}{\sigma} \tag{1b}$$

$$\dot{\underset{\sim}{v}} = \frac{\partial}{\partial \underset{\sim}{\sigma}}\left\{\frac{\phi^{n+1}(\underset{\sim}{\sigma})}{n+1}\right\} f(t, \theta(t)) \tag{1c}$$

where $\underset{\sim}{\epsilon}$, $\underset{\sim}{e}$, and $\underset{\sim}{v}$ denote the total, elastic and viscous strains respectively. The elastic strain component $\underset{\sim}{e}$ possesses a positive definite complementary strain energy density

$$E(\underset{\sim}{\sigma}) = \frac{1}{2} C_{ijkl}\sigma_{ij}\sigma_{kl} \tag{2}$$

and ϕ denotes a homogeneous function of degree one in the components of the stress tensor $\underline{\sigma}$. The rate of creep energy dissipation D^c is given by

$$\dot{D}_C(\underline{\sigma}) = \underline{\sigma}\dot{\underline{v}} = \phi^{n+1}(\underline{\sigma}) f(t, \theta(t)) \tag{3}$$

Here $\theta(t)$ denotes the temperature.

Consider a body of this material which is subject to surface forces $\underline{P}(\underline{x}, t)$, body forces $\underline{F}(\underline{x}, t)$ and a thermal field $\theta(\underline{x}, t)$. Part of the surface S, which we denote by S_u is subject to a history of surface displacement $\underline{U}(\underline{x}, t)$. Within the volume, inelastic strains $\underline{\Delta}(\underline{x}, t)$ are induced by either the volumetric change due to θ or by some other externally applied agency. We assume that all the externally applied agencies, $\underline{P}$, $\underline{u}$, θ and $\underline{\Delta}$ vary periodically with time with periodicity Δt. It was shown in [3] that the resulting stress field $\underline{\sigma}(\underline{x}, t)$ and strain rate field $\dot{\underline{\epsilon}}(x, t)$ possess periodicity with period Δt as an asymptotic state as the time from initial loading approaches infinity. Thus the well known stationary state response of a body subject to constant loading (see for example[1]) is replaced by a stationary cyclic state for a body subject to cyclic loading. Although a more general situation may be carried through, in this paper we confine attention to this cyclic state as it appears to be the problem of greatest practical significance, as the additional displacements occurring during the transition from the initial elastic state to the cyclic state is generally small.

In [4], an upper bound on the displacement of a body during a cycle of the cyclic state was derived:

$$\int_{S_T} \underline{T}\{\underline{u}(\Delta t) - \underline{u}(0)\}\, dS \leq \frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1} \int_V \int_0^{\Delta t} \dot{D}^c(\hat{\underline{\sigma}} + \hat{\underline{\sigma}}^T + \bar{\underline{\rho}})\, dt\, dV \tag{4}$$

Here $\hat{\underline{\sigma}}(t)$ denotes the instantaneous linear elastic solution to the problem, and $\bar{\underline{\rho}}$ denotes a time constant arbitrary equilibrium residual state of stress. Thus $\bar{\underline{\rho}}$ is in equilibrium with zero surface tractions on S_T. Surface tractions $\underline{T}$ denote some "dummy" loads which would give rise to a linear elastic solution $\hat{\underline{\sigma}}^T$ for the same body geometric and elastic material properties but with zero applied displacements on S_u and zero applied inelastic strain $\underline{\Delta}$ within V. The loads $\underline{T}$ remain constant during the period $0 < t < \Delta t$. The inequality (4) may provide a point displacement bound if $\underline{T}$ is identified with a point load, or an average displacement bound when $\underline{T}$ becomes a distributed load. It may be noted that the elastic stress distribution $\hat{\underline{\sigma}}^T$ may be replaced by arbitrary equilibrium state of stress in equilibrium with $\underline{T}$, which we denote by $\underline{\sigma}^T$. As both $\underline{T}$ and $\bar{\underline{\rho}}$ remain constant in time we may write

$$\hat{\underline{\sigma}} + \hat{\underline{\sigma}}^T + \bar{\underline{\rho}} = \hat{\underline{\sigma}} + \underline{\sigma}^T + \bar{\underline{\rho}}' \tag{5}$$

where $\bar{\underline{\rho}}' = \bar{\underline{\rho}} + (\hat{\underline{\sigma}}^T - \underline{\sigma}^T)$ denotes another arbitrary residual stress field.

In general, to compute an upper displacement bound we write

$$\underline{T} = T\underline{t}(\underline{x}) \tag{6}$$

where $\underline{t}$ denotes a load of unit intensity at a point $\underline{x}$, and attempt to find the value of $\bar{\underline{\rho}}$ and the value of T which will provide the optimal value of the upper bound

$$\int_{S_T} \underline{t}\{\underline{u}(\Delta t) - \underline{u}(0)\}\, dS \leq \frac{1}{T}\frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1} \int_V \int_0^{\Delta t} \dot{D}^c(\hat{\underline{\sigma}} + \hat{\underline{\sigma}}' T + \bar{\underline{\rho}})\, dt\, dV. \tag{7}$$

3. WORK BOUNDS

Before considering the central problem of the paper we review some results[5, 3] for work bounds. These bounds take the form

$$\int_V \int_0^{\Delta t} \dot{D}^c(\underline{\sigma}^s)\, dt\, dV \leq \int_V \int_0^{\Delta t} \underline{\sigma}(\dot{\underline{\epsilon}} - \dot{\underline{\Delta}})\, dt\, dV \leq \int_V \int_0^{\Delta t} \dot{D}^c(\hat{\underline{\sigma}} + \bar{\underline{\rho}})\, dt\, dV \tag{8}$$

In (8) the stress field $\hat{\underset{\sim}{\sigma}}$ and $\bar{\underset{\sim}{\rho}}$ have the meaning of the quantities contained in inequality (4). The stress field $\underset{\sim}{\sigma}^s(t)$ denotes the steady state solution for the loads at a time t which may be obtained by ignoring the elastic strain $\underset{\sim}{e}$ and the inelastic volume change $\underset{\sim}{\Delta}$.

It was shown in[5] that the optimal value of the right hand side of (8) was given by the (unique) residual stress field $\bar{\underset{\sim}{\rho}}$ which gave rise to an accumulated creep strain over a complete cycle

$$\Delta^u \underset{\sim}{v}(0, \Delta t) = \int_0^{\Delta t} \dot{\underset{\sim}{v}}(\hat{\underset{\sim}{\sigma}} + \bar{\underset{\sim}{\rho}})\, \mathrm{d}t \tag{9}$$

which was internally compatible and compatible with zero displacements on S_u. Here $\dot{\underset{\sim}{v}}(\hat{\underset{\sim}{\sigma}} + \bar{\underset{\sim}{\rho}})$ denotes the creep strain rate computed from the creep relationship (1c). We find that this result forms the basis for the discussion of inequality (7).

We note further that the creep rate arising from $\underset{\sim}{\sigma}^s$ in the lower work bound (8) is also compatible and therefore accumulates over a cycle creep strains

$$\Delta^L \underset{\sim}{v} = \int_0^{\Delta t} \dot{\underset{\sim}{v}}(\underset{\sim}{\sigma}^s)\, \mathrm{d}t \tag{10}$$

which are compatible. From both $\Delta^u \underset{\sim}{v}$ and $\Delta^h \underset{\sim}{v}$ we may derive displacement increments $\Delta \underset{\sim}{u}^u$ and $\Delta \underset{\sim}{u}^L$.

The lower bound and the optimal upper bound of inequalities (8) may be interpreted as two particular solutions to the problem which occurs when the time scale of loading is very long and very short compared with the characteristic material time[5, 3].

4. THE OPTIMAL UPPER DISPLACEMENT BOUND

We first assume that T has an assigned value and find the condition that the bound (7) will be optimal with respect to $\bar{\underset{\sim}{\rho}}$. The condition is immediately apparent from the result for the upper work bound (8). And we may state:

The minimum upper bound (7) for fixed T is given by $\bar{\underset{\sim}{\rho}} = \bar{\underset{\sim}{\rho}}^*$ *for which the strain field*

$$\Delta \bar{\underset{\sim}{v}} = \int_0^{\Delta t} \dot{\underset{\sim}{v}}(\hat{\underset{\sim}{\sigma}} + \hat{\underset{\sim}{\sigma}}^T + \bar{\underset{\sim}{\rho}}^*)\, \mathrm{d}t \tag{11}$$

shall be internally compatible and give rise to zero displacements on S_u.

Thus the minimum value of the volume integral in (7) with respect to $\bar{\underset{\sim}{\rho}}$ is identical to the minimum value of the upper work bound (8) for the case when the loading $\underset{\sim}{P}$ is augmented by the dummy surface traction $\underset{\sim}{T}$. In[3] and[5] it was shown that this upper work bound (8) was approached within the body when the cycle time was very short compared with a characteristic time of the material, and we may therefore identify the optimal values with a state that may be achieved within the body by suitable choice of time scales, for this augmented situation.

We now assume that $\bar{\underset{\sim}{\rho}}$ possesses this optimal value $\bar{\underset{\sim}{\rho}}^*$ and find the value of T for which the bound (7) has its minimum value. Noting that $\bar{\underset{\sim}{\rho}}^*$ varies with T, stationary values of the bound with respect to T are given by

$$-\frac{1}{T^2}\int_V \int_0^{\Delta t} \dot{D}^c \, \mathrm{d}t \, \mathrm{d}V + \frac{1}{T}\int_V \int_0^{\Delta t} \left(\hat{\underset{\sim}{\sigma}}^t + \frac{\partial \bar{\underset{\sim}{\rho}}^*}{\partial T}\right)\left(\frac{\partial \dot{D}^c}{\partial \underset{\sim}{\sigma}}\right) \mathrm{d}t \, \mathrm{d}V = 0 \tag{12}$$

on noting that

$$\frac{\partial \dot{D}^c}{\partial \underset{\sim}{\sigma}} = (n+1)\dot{\underset{\sim}{v}}$$

We see that (12) may be written as

$$\int_V \int_0^{\Delta t} \dot{D}^c \, \mathrm{d}t \, \mathrm{d}V = (n+1)\int_V T\hat{\underset{\sim}{\sigma}}^t \Delta \bar{\underset{\sim}{v}} \, \mathrm{d}V + (n+1)\int_V T\frac{\partial \bar{\underset{\sim}{\rho}}^*}{\partial \underset{\sim}{\sigma}} \Delta \bar{\underset{\sim}{v}} \, \mathrm{d}V \tag{13}$$

As $(\partial \bar{\rho}^*/\partial T)$ denotes a constant residual stress field and as $\Delta\bar{\underline{v}}$ is compatible with zero displacement on S_u, the second integral of (11) becomes zero. By the principle of virtual work we therefore obtain

$$\int_V \int_0^{\Delta t} \dot{D}^c \, dt \, dV = (n+1) \int_{S_T} T\underline{t}\Delta\bar{\underline{u}} \, dS \tag{14}$$

where $\Delta\bar{\underline{u}}$ denotes the displacements compatible with $\Delta\bar{\underline{v}}$. By further differentiating with respect to T it may be shown that (10) and therefore (12) provides the absolute minimum with respect to T.

Substituting (14) in to the bound (7) we obtain

$$\int_{S_T} \underline{t}\Delta\underline{u} \, dS \leq \left(\frac{n}{n+1}\right)^n \int_{S_T} \underline{t}\Delta\bar{\underline{u}} \, dS \tag{15}$$

where $\Delta\bar{\underline{u}}$ is computed from the value of T given by (12).

In the form (14) the condition does not lend itself to a simple interpretation. We may remember however that the strains and displacements $\Delta\bar{\underline{v}}$ and $\Delta\bar{\underline{u}}$ may be interpreted as the analytic solution of the stated continuum problem with the loads augmented by the dummy loads T and the cycle time Δt infinitesimally small. If we ignore $\underline{\Delta}$, we may use the property to relate $\underline{P}$ and T directly. By the principle of virtual work we may write (formally):

$$\int_V \int_0^{\Delta t} \dot{D}^c \, dt \, dV = \int_{S_T} \int_0^{\Delta t} \underline{P}\dot{\bar{u}} \, dt \, dS + \int_{S_T} \underline{T}\Delta\bar{\underline{u}} \, dS,$$

and (12) becomes

$$\int_{S_T} \int_0^{\Delta t} \underline{P}\dot{\bar{u}} \, dt \, dS = n \int_{S_T} \underline{T}\Delta\bar{\underline{u}} \, dS \tag{16}$$

Thus the optimal value of T is given by the condition that the ratio of the work done by $\underline{P}$ on the solution $\bar{\underline{u}}$ to that done by T shall be n. When $\underline{T}$ acts in the direction and position of $\underline{P}$ we see that $\underline{T}$ will be the order of magnitude $(1/n)\underline{P}$.

If $\underline{P}$ remains constant in time and $\underline{P} = P\underline{t}$ the equation (14) yields the optimal value

$$T = P/n$$

and the displacements $\Delta\bar{\underline{u}}$ are derived from the stress field

$$\hat{\underline{\sigma}} + \hat{\underline{\sigma}}^T + \bar{\underline{\rho}}^* = \frac{n+1}{n} \underline{\sigma}^s$$

where $\underline{\sigma}^s$ denotes the steady state solution for load P. The bound (15) then becomes

$$\int_{S_T} \underline{t}\Delta\underline{u} \, dS \leq \int_{S_T} \underline{t}\Delta\underline{u}^s \, dS \tag{17}$$

where $\Delta\underline{u}^s$ denotes the increment of steady state displacement over the cycle time. It is clear the equality holds in (17).

We see therefore that the optimal value of the bound involves solution of a continuum problem for $\bar{\underline{\rho}}^*$ and a search for the value of T for which equation (14) is satisfied. In the next section we investigate the procedure for the simple problem of a propped cantilever subject to a point load $P(t)$. We find that even for widely varying $P(t)$ for $n \geq 3$ a near optimal bound may be found by assuming

$$T = \frac{1}{n} \max\{P(t)\} \tag{18}$$

It is found that this equation provides an adequate approximation even when the point of application of T is some distance from that of $P(t)$. This result is not entirely unexpected as the bound may be expected to be fairly insensitive to the value of T near its minimum value.

If we include $\underset{-}{\Delta}$ equation (16) becomes

$$\int_{S_T}\int_0^{\Delta t} \underset{-}{P}\dot{\underset{-}{u}}\,dt\,dS = \int_V \int_0^{\Delta t} (\underset{-}{\hat{\sigma}} + T\underset{-}{\hat{\sigma}}^t + \underset{-}{\bar{\rho}}^*)\dot{\underset{-}{\Delta}}\,dt\,dV + n\int_{S_T} \underset{-}{T}\Delta\underset{-}{u}\,dS$$

The additional term corresponds to the work done by $\dot{\underset{-}{\Delta}}$ against the stress field due to $\underset{-}{P}$ and T.

5. AN EXAMPLE

We consider the problem of a cantilever of length l, simply supported at one end ($x = l$) and encastre at the other end ($x = 0$). A lateral load $P(t)$ acts at $x = l/4$ (Fig. 1).

We will assume that the curvature $\kappa(t)$ of the beam is given in terms of the moment M by

$$\begin{aligned}\dot{\kappa} &= \dot{\kappa}_e + \dot{\kappa}_c \\ &= \frac{\dot{M}}{EI} + kM^n\end{aligned}$$

Here E, I and n have the usual meanings and k denotes a constant. We will not consider the conditions under which such an approximation is appropriate, as such questions lie outside the scope of this paper.

The structure possesses a single redundancy which may be taken as the reaction at $x = l$. The most general moment distribution for the load P is given by

$$M(x) = Pm(x, l/4) - R\bar{m}(x)$$

where $m(x, y)$ denotes the moment distribution at x due to a unit load at y assuming zero moment at $x = 0$ and $x = l$, and $\bar{m}(x)$ denotes the moment due to a unit load at $x = l$.

For the linear elastic case we find $R = +(21/128)P$ and

$$\hat{M}(x, t) = P(t)\left\{m(x, l/4) - \frac{21}{128}\bar{m}(x)\right\}$$

A general residual bending moment is given by

$$\bar{M} = \bar{R}\bar{m}(x)$$

and the upper bound (7) achieves the form

$$\Delta u(y) \leq \frac{1}{T}\frac{1}{n}\left(\frac{n}{n+1}\right)^{n+1}\int_0^l\int_0^{\Delta t} k(\hat{M} + Tm(x, y) + \bar{R}\bar{m}(x))^n\,dt\,dx \qquad (19)$$

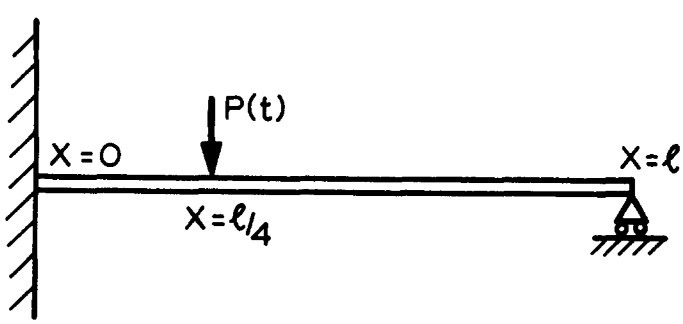

Fig. 1.

where $\Delta u(y)$ denotes the increment of lateral deflection of the beam during the time interval $t = 0$ to $t = \Delta t$.

For fixed value of T the optimal value of $\bar{R}$ is given by the condition that κ_c accumulated over a cycle due to both P and T

$$\Delta\bar{\kappa} = \int_0^{\Delta T} k(\hat{M} + Tm(x, y) + \bar{R}\bar{m}(x))^n \, dt \tag{20}$$

shall be compatible with zero vertical displacement at $x = l$. By the principle of virtual work, $\bar{R}$ is therefore given by the solution of the equation

$$\int_0^l \Delta\bar{\kappa}(x)\, \bar{m}(x)\, dx = 0. \tag{21}$$

Equation (21) was solved numerically by Newton–Raphson for $\bar{R}$ after suitable non-dimensionalizing. Integration was performed by Simpson's Rule and care was taken to ensure that the significant figures quoted in the final results of the computation contained no errors due to numerical integration.

The optimal value of T is given by equation (14) which becomes

$$\frac{1}{T}\int_0^l \int_0^{\Delta t} \dot{D}^c \, dt \, dx = \frac{1}{T}\int_0^l \int_0^{\Delta t} k(\hat{M} + Tm(x, y) + \bar{M})^{n+1} \, dt \, dl = (n+1)\Delta\bar{u}(y) \tag{22}$$

The value of $\Delta\bar{u}(y)$ was obtained by virtual work from $\Delta\bar{\kappa}$:

$$\Delta\bar{u}(y) = \int_0^y \Delta\bar{\kappa}(x) m^*(y, x)\, dx \tag{23}$$

where $m^*(y, x)$ denotes the moment at x due to a unit vertical load at $x = y$ assuming no support at $x = l$.

The loading history $P(t)$ was chosen of the form shown in Fig. 2. Load P was maintained for $\Delta t/2$ followed by a load λP for $\Delta t/2$. Displacement bounds were computed at the position of the $x = l/4$ and at the centre of the beam, $x = l/2$ for a range of values of n and λ. The computational sequence was as follows. In each case, for a range of values of T the optimal value of $\bar{R}$ was computed by solving equation (21) by Newton–Raphson. In each case $\Delta\bar{u}(y)$ and the left hand side of (22) were computed and the value of T for which (22) was satisfied was found. The optimal bound was then given, from equation (15) by $\Delta u \leq (n/n+1)^n \Delta\bar{u}$. The results of such

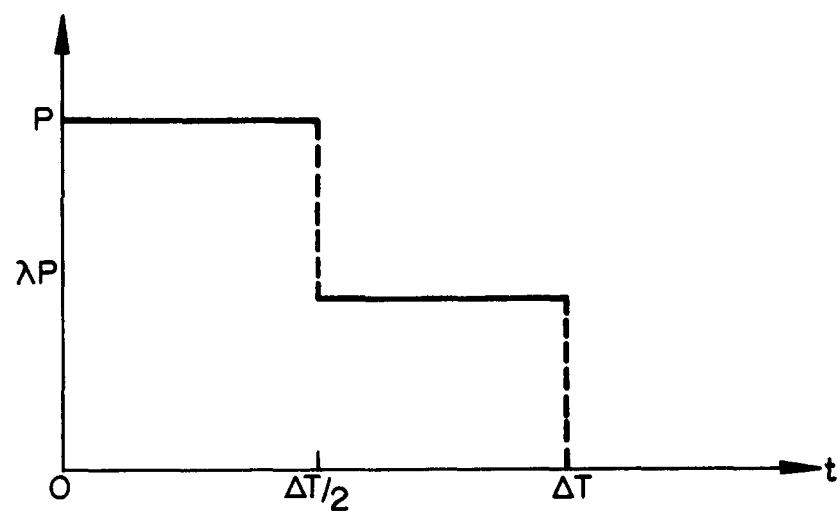

Fig. 2.

a sequence are shown in Fig. 3 for the case $n = 3$ and $\lambda = 1$ (constant load P). The point of intersection of the two sides of (22) occurs at $T/p = 1/n$ and it can be seen that the value of the left hand side of (22), and therefore the upper bound is insensitive to the value of T near this value.

The optimal bounds for $n = 3$ and $n = 7$ are shown in Table 1 and 2, for values of $\lambda = 1, 0{\cdot}5, 0{\cdot}0$ and $-0{\cdot}5$. The upper and lower work bounds (8) were also computed and the associated deflection $\Delta^u \underline{u}$ of the upper bound is included in the tables. All the values have been normalized with respect to $\Delta \underline{u}^L = \Delta u^s$, the displacements which would occur if purely viscous behaviour was assumed. The values of the upper bound were also computed assuming equation (18)

$$T = \frac{1}{n} \max\{P(t)\} \tag{18}$$

There are several striking features of the results. For both values of n and for $\lambda = 1{\cdot}0, 0{\cdot}5$ and $0{\cdot}0$ the upper bound and $\Delta u^u/\Delta u^L$ lie close to unity and the presence of elastic strains are clearly small. For the case $\lambda = -0{\cdot}5$ the displacement prediction $\Delta u^u/\Delta u^L$ of the work bounds deviate appreciably from unity and indicate a smaller value for the upper work bound, which corresponds to rapid cycling. In all cases the optimal upper bound lies tolerably close to the larger of the two predictions Δu^u and Δu^L. Further, the non-optimal bound obtained from (18) differs only slight from the optimal value.

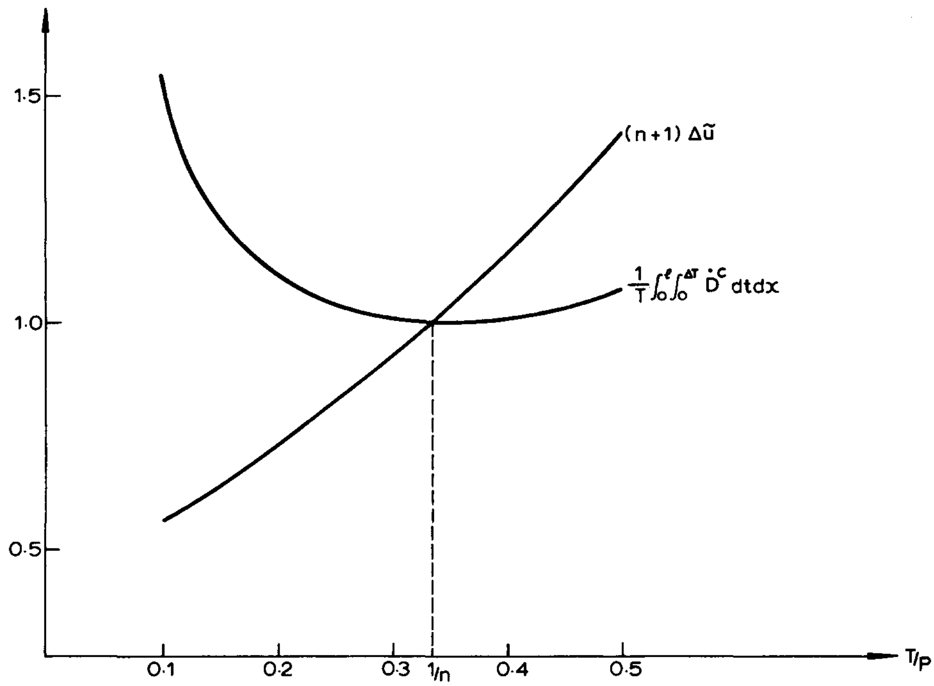

Fig. 3. Variation of upper bound with $\underline{T}$ near the optimal value and the variation of equation (18) with $\underline{T}$.

Table 1. $n = 3$

			(Upper bound)	Δu^L	
	y/l	$\Delta u^u/\Delta u^L$	Optimal	$\frac{T}{\max P} = \frac{1}{n}$	T/P
$\lambda = 1$	0·25	1·0000	1·0000	1·0000	0·333
	0·5	1·0000	1·1240	1·2045	0·227
$\lambda = 0{\cdot}5$	0·25	1·0185	1·0373	1·0408	0·304
	0·5	1·0178	1·1733	1·3577	0·205
$\lambda = 0{\cdot}0$	0·25	0·9989	1·0205	1·0208	0·324
	0·5	0·9948	1·1561	1·2688	0·216
$\lambda = -0{\cdot}5$	0·25	0·8561	1·1663	1·1685	0·356
	0·5	0·8514	1·3212	1·4005	0·242

Table 2. $n = 7$

	y/ℓ	$\Delta u^u/\Delta u^L$	(UPPER BOUND) / Δu^L OPTIMAL	(UPPER BOUND) / Δu^L $\frac{T}{\max P} = \frac{1}{n}$	OPTIMAL T/P
$\lambda = 1$	0·25	1·000	1·000	1·000	0·143
	0·5	1·000	1·1104	1·1179	0·123
$\lambda = 0.5$	0·25	1·0256	1·0280	1·0281	0·141
	0·5	1·0315	1·1453	1·1638	0·120
$\lambda = 0.0$	0·25	0·9991	1·0008	1·0008	0·143
	0·5	0·9988	1·1055	1·1194	0·123
$\lambda = -0.5$	0·25	0·6114	1·1147	1·1406	0·174
	0·5	0·5994	1·2530	1·2550	0·150

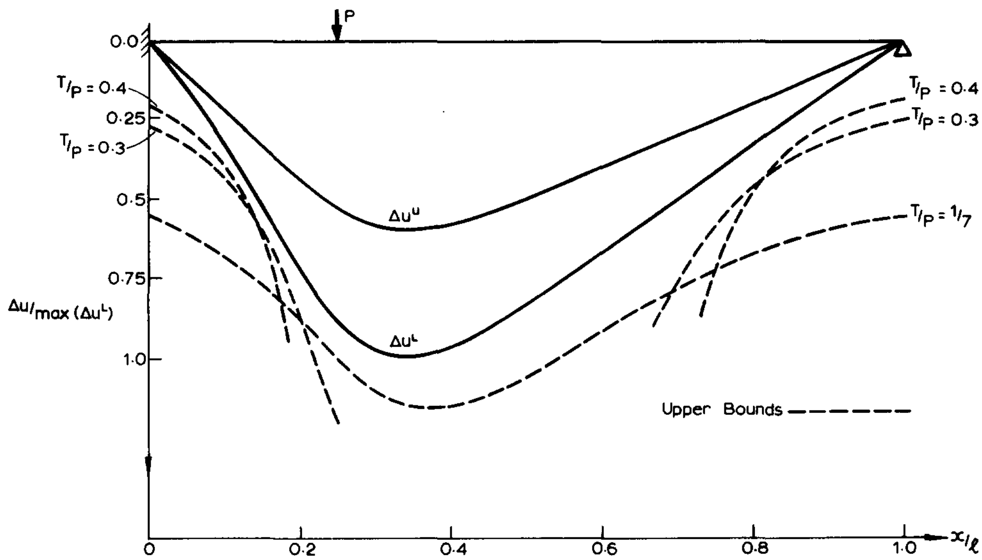

Fig. 4. Displacement bounds for $n = 7$, $\lambda = -0{\cdot}5$.

In Fig. 4 displacement bounds were computed at a sequence of stations along the length of the cantilever for $n = 7$ and $\lambda = -0{\cdot}5$. A fixed value of T was chosen and optimal bounds with respect to $\bar{R}$ were evaluated for a sequence of points, producing a bounding contour which became an optimal bound at some point along the beam. A sequence of such bounding contours combine to form a reasonable bound for all points of the beam. The calculations are compared with Δu^u and Δu^L.

It is interesting to note that in this example the purely viscous solution Δu^L would, in all cases, provide an acceptable and almost certainly conservative answer. The effect of thermal loading has yet to be explored any may well disturb this simple picture.

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